Lecture 24.

Invariance Properties of n-D Lebesgue $m = m^n$.

Thm 1. (Translation Invariance). Let $\tau_a(x) = x + a$.

(i) $E \in \mathcal{L}^n \Rightarrow \tau_a(E) \in \mathcal{L}^n$ and $m(\tau_a(E)) = m(E)$.

(ii) f $\mathcal{L}^n$-meas. $\Rightarrow$ $f \circ \tau_a$ $\mathcal{L}^n$ meas. and if $f \in L^+$ or L^1, then

$$\int f \, dm = \int (f \circ \tau_a) \, dm.$$

Pf. (i) τ_a is a homeomorphism (cont. w/ cont. inverse) and $(\tau_a)^{-1} = \tau_{-a}$. Thus, $(\tau_{-a})^{-1}$ is $(\mathcal{B}_{\mathbb{R}^n}, \mathcal{B}_{\mathbb{R}^n})$-meas. $\Rightarrow$ $\tau_a(B) = (\tau_{-a})^{-1}(B)$ is in $\mathcal{B}_{\mathbb{R}^n}$ for every $B \in \mathcal{B}_{\mathbb{R}^n}$. Moreover, for any rectangle $B_1 \times B_2 \times \dots \times B_n$ w/ $B_j \in \mathcal{B}_{\mathbb{R}}$, $m(B_1 \times \dots \times B_n) = m(B_1) \dots m(B_n)$ by 1-D result (which follows by using the uniqueness of extension of premeas. algebra of intervals to Borel sets; $\mathbb{R}$ is σ-finite).

Then, since $\mathbb{R}^n$ is σ-finite for m, the uniqueness of extension of premeasure $\Rightarrow$ $m(\tau_a(B)) = m(B)$, $\forall B \in \mathcal{B}_{\mathbb{R}^n}$.

Now, if $E \in \mathcal{L}^n$, then $E = B \cup A$, where $B \in \mathcal{B}_{\mathbb{R}^n}$ and $A \subseteq N \in \mathcal{B}_{\mathbb{R}^n}$ w/ $m(N) = 0$. $\Rightarrow$ $\tau_a(E) = \tau_a(B) \cup \tau_a(A)$, $\tau_a(B) \in \mathcal{B}_{\mathbb{R}^n}$ and $\tau_a(A) \subseteq \tau_a(N)$, $m(\tau_a(N)) = 0$ by above $\Rightarrow$ $\tau_a(E) \in \mathcal{L}^n$, and $m(\tau_a(E)) = m(\tau_a(B)) = m(B) = m(E)$.

(ii) $\mathbb{R}^n \xrightarrow[\tau_a]{} \mathbb{R}^n \xrightarrow[f]{} \mathbb{C}$ (or $\overline{\mathbb{R}}$). If $E \in \mathcal{B}_{\mathbb{C}}$, then $f^{-1}(E) \in \mathcal{L}^n \Rightarrow f^{-1}(E) = B \cup A$ as above $\Rightarrow$ $(f \circ \tau_a)^{-1}(E) = (\tau_a)^{-1}(B) \cup (\tau_a)^{-1}(A) = \tau_{-a}(B) \cup \tau_{-a}(A)$. By (i) $\tau_{-a}(B) \in \mathcal{B}_{\mathbb{R}^n}$ and $m(\tau_{-a}(A)) = 0$ $\Rightarrow$ $(f \circ \tau_a)^{-1}(E) \in \mathcal{L}^n$, i.e., $f \circ \tau_a$ $\mathcal{L}^n$-meas.

Now, $\int (f\circ\tau_a)\,dm = \int f\,dm$ for $f=\chi_E$
$\Rightarrow$ for $f=\varphi$ simple $\Rightarrow$ for all
$f\in L^+$ or L^1 by MCT and DCT
since we can approximate $\varphi_n \to f$ a.e.
w/ $|\varphi_n| \nearrow |f|$. $\boxtimes$

Recall that a linear map $T: \mathbb{R}^n \to \mathbb{R}^n$ is given by a matrix $T = (T)_{ij}$ s.t

$$y = T(x) \Leftrightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

- We have T invertible, denoted $GL(n,\mathbb{R})$, if $\det T \neq 0$.

- Any $T \in GL(n,\mathbb{R})$ can be expressed as $T = S_1 \circ \dots \circ S_m$, where S_m are elementary matrices, i.e., one of 3 types:

(I) $S = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \Rightarrow \det S = c$

(II) $S = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \Rightarrow \det S = -1$

(III) $S = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & c & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \Rightarrow \det S = 1$

<u>Thm 2.</u> Let $T \in GL(n, \mathbb{R})$

(i) $E \in \mathcal{L}^n \Rightarrow T(E) \in \mathcal{L}^n$ and

$$m(T(E)) = |\det T| \, m(E).$$

(ii) f is $\mathcal{L}^n$-meas. $\Rightarrow (f \circ T)$ is $\mathcal{L}^n$-meas.

If $f \in L^+$ or L^1, then

$$\int f \, dm = |\det T| \int (f \circ T) \, dm.$$